Intermittency, Fractals, and β-model

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I. INTRODUCTION

An essential assumption of Kolmogorov 1941 theory is that eddies of any generation are space filling at inertial-range scales. It assumes that energy transfer to small scale fluctuations is uniform, i.e. $\bar{\epsilon}$ is independent of space and time. However, the energy dissipation rate per unit mass of a turbulent fluid, ϵ, is a random function of x and t, which fluctuates with the random velocity field $v(x,t)$. These small scale fluctuations are observed to be highly nonuniform in space and time, which is denoted as *intermittent*. The word *intermittency* suggests the small scale turbulence tend to concentrate into small and isolated regions or time intervals surrounded by smoother large scale flows. As the scale decreases, or the Reynolds number increases, the intermittency effects should become more and more important. In this sense the K41 theory is incomplete.

Geometrically speaking, the intermittency indicates that the dimension of the dissipative structures is smaller than the space dimension, i.e. $D_{\text{eff}} < D_{\text{space}}$, which reminds us of the fractals. In fact, fractals have been used in intermittency models. In following sections, we will briefly review the properties of fractals first, and then introduce the fractal model of intermittency.

II. FRACTALS

A fractal is an object or a mathematical set that exhibits a self-similar pattern. It is characterized by a fractal dimension, which is a measure of the space-filling capacity of a pattern and tells how a fractal scales differently from the space it is embedded in. A fractal dimension does not have to be an integer.

In practice, the fractal dimension can be approximated by the *box-counting dimension* [1]. Imagine the fractal lying on an evenly spaced grid, and count how many boxes are required to cover the set. Then the box-counting dimension is calculated by seeing how this number changes as we make the grid finer, i.e.

$$D_0 = \lim_{\epsilon\to 0} \frac{\ln N(\epsilon)}{\ln\left({}^{1}\!/\!_{\epsilon}\right)},$$

where ϵ is the length of each grid and $N(\epsilon)$ is the number of boxes of side length ϵ. Roughly speaking, this suggests the number of boxes needed to cover the set increases with ϵ in a power law fashion $N \sim \epsilon^{-D_0}$. Let us see some examples.

Example 1. Dimension of finite number points. Box number N is finite and independent of ϵ, then

$$D_0 = \lim_{\epsilon\to 0} \frac{\ln N}{\ln\left({}^1\!/\!_\epsilon\right)} = 0.$$

Example 2. Dimension of a smooth curve segment of length l. The box number is $N = {}^l\!/\!_\epsilon$, then

$$D_0 = \lim_{\epsilon\to 0} \frac{\ln {}^l\!/\!_\epsilon}{\ln\left({}^1\!/\!_\epsilon\right)} = \lim_{\epsilon\to 0}\left(1 - \frac{\ln l}{\ln \epsilon}\right) = 1.$$

Example 3. Dimension of an area A. The box number is $N = {}^A\!/\!_{\epsilon^2}$, then

$$D_0 = \lim_{\epsilon\to 0} \frac{\ln {}^A\!/\!_{\epsilon^2}}{\ln\left({}^1\!/\!_\epsilon\right)} = \lim_{\epsilon\to 0}\left(2 - \frac{\ln A}{\ln \epsilon}\right) = 2.$$

These three examples are trivial cases of smooth space with integer dimension. The illustrations of box-counting $N(\epsilon)$ are shown in figure 1. Now let us check some nontrivial cases.

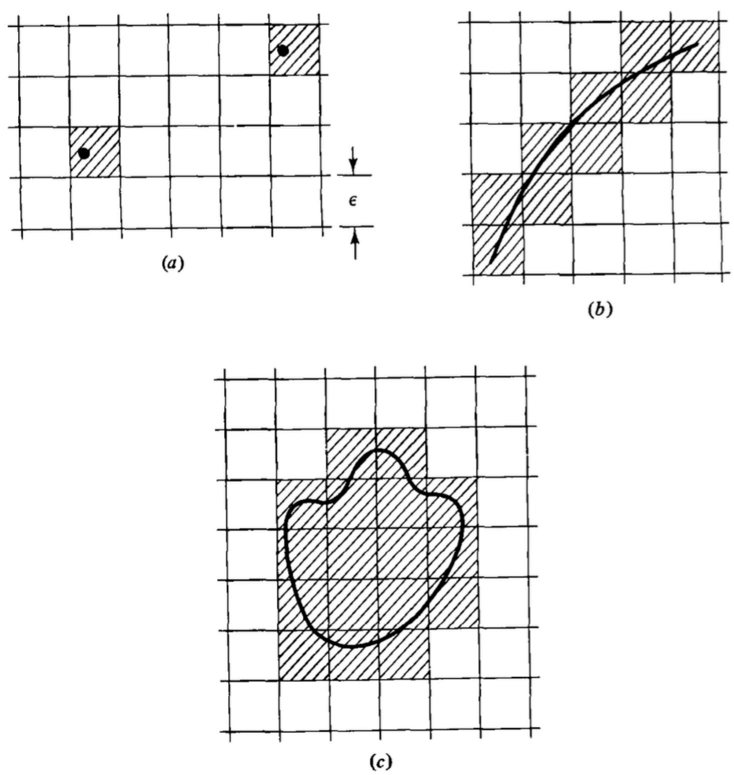

Figure 1. Illustrations of box-counting $N(\epsilon)$ for sets consisting (a) finite points, (b) a smooth curve segment, and (c) the area enclosed by a curve.

Example 4. Dimension of the Cantor ternary set, which is built by removing deleting the open middle third from each of a set of line segments repeatedly, as shown in figure 2.

1. One starts by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the interval $[0, 1]$, leaving two line segments: $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

2. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

3. This process is continued ad infinitum.

The Cantor ternary set is *self-similar* in the sense that smaller pieces of it reproduce the entire set upon magnification.

In this case it is better to use iteration number n instead of the box length ϵ for box-counting. We assume the length of the set is 1 without loss of generality, then at n iteration the length of box is $({}^1\!/\!{}_3)^n$, box number $N = (3-1)^n = 2^n$, the dimension

$$D_0 = \lim_{n\to\infty} \frac{\ln 2^n}{\ln\left(\frac{1}{({}^1\!/\!{}_3)^n}\right)} = \frac{\ln 2}{\ln 3} \approx 0.631,$$

which is a non-integer between zero and one, indicating the set is a fractal embedded in the one-dimension space.

Figure 2. A Cantor ternary set in seven iterations.

Example 5. Dimension of Koch curve, also known as the "coastline curve", as illustrated in figure 3. It can be constructed by starting with an line segment of length 1, then recursively altering as follows:

1. divide the line segment into three segments of equal length.

2. draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.

3. remove the line segment that is the base of the triangle from step 2.

After each iteration, the number of sides of the Koch curve increases by a factor of 4, so the number of sides after n iterations is $N_n = N_{n-1} \cdot 4 = 4^n$, and the length of each side is $(1/3)^n$. The dimension is then

$$D_0 = \lim_{n\to\infty} \frac{\ln 4^n}{\ln\left(\frac{1}{(1/3)^n}\right)} = 2\frac{\ln 2}{\ln 3} \approx 1.263,$$

which is between one and two. So Koch curve is a fractal embedded in two-dimension space, and it is *thickened* from an one-dimension line segment. As the resolution increases, $N(\epsilon) \sim \epsilon^{-D_0}$. It is akin to many natural fractals like coastlines, snowflakes and mountain ranges.

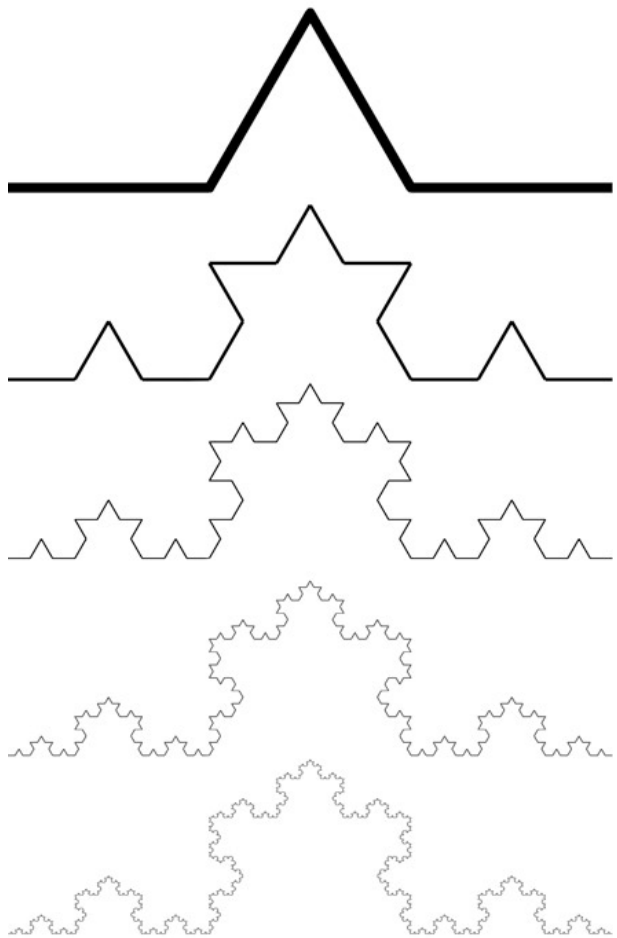

Figure 3. Koch curve.

As mentioned above, a fractal is self-similar, i.e. repeats the same or approximately similar pattern at every scale. The fractal dimension also deviates from the dimension of the embedding space. These two properties make fractals natural candidates to describe:

- intermittent dissipation events,
- geometry of dissipative structures in intermittent turbulence,

since in cascades the dissipative structures do not filling the space uniformly. The inclusion of fractals can give an intermittency correction to the K41 spectrum.

III. THE β-MODEL

Experimental evidence has shown that the small-scale structures of turbulence become less and less space filling as the scale size decreases. This patchiness of the small scales can be understood by a simple vortex-stretching argument. From the vorticity equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\omega = \nabla \times (\mathbf{v} \times \omega) + \nu \nabla^2 \omega \\
&\frac{d}{dt}\omega = \omega \cdot \nabla \mathbf{v} + \nu \nabla^2 \omega \\
&|\omega| \sim |\nabla \mathbf{v}|
\end{aligned}$$

we obtain

$$\frac{d}{dt}|\omega| \sim \left|\omega^2\right|.$$

Hence it must expected that the vorticity/enstrophy will grow explosively through vortex-stretching process, in which the turbulent activities become more and more concentrated. Therefore, there should be intermittency corrections to K41 theory of the inertial-range.

The β-model is the simplest fractal intermittency model which incorporates an intermittency modification to the K41 model [2, 3]. The basic idea (shown in figure 4) is that at each step of cascades, the number of "children" of a given "parent" eddy is chosen such that the fraction of volume occupied is decreased by a factor β ($0 < \beta < 1$). The factor β is an adjustable parameter of the model. Otherwise, nothing is changed from the K41 theory.

Let us now define β-model. At each step of the cascade process, any n-eddy of size $l_n = l_{n-1}/2 = l_0 2^{-n}$ produces N $(n+1)$-eddies on average. It needs 2^3 "children" eddies to fill the full space at each step. If the largest eddies are space filling, after n iterations only a fraction

$$\beta_n = \beta^n = \left(\frac{N}{2^3}\right)^n$$

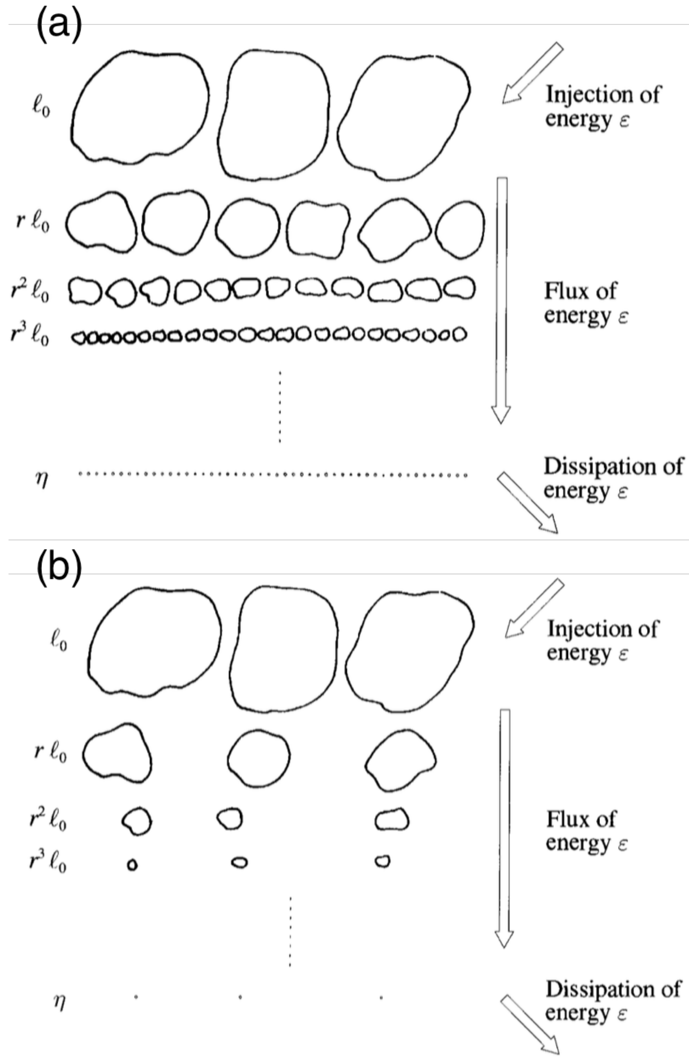

Figure 4. (a) Kolmogorov 41 cascade, at each step the eddies are full space-filling. (b) β-model cascade, at each step the eddies become less and less space-filling.

of the space will be occupied by active fluid. Now we denote $N \equiv 2^D$, where the D is the fractal dimension, then it is straightforward to work out how β-model modifies K41:

$$
\begin{aligned}
\beta &= \left(2^{D-3}\right)^n = \left(\frac{l_0}{l_n}\right)^{D-3}, \\
E_n &\sim \beta_n v_n^2, \\
t_n &= \frac{l_n}{v_n}, \\
\epsilon_n &\sim \frac{E_n}{t_n} \sim \bar{\epsilon}.
\end{aligned}
$$

Combining these equations, we obtain

$$v_n \sim \bar{\epsilon}^{\frac{1}{3}} l_n^{\frac{1}{3}} \left(\frac{l_n}{l_0}\right)^{-\frac{1}{3}(3-D)},$$

$$t_n \sim \bar{\epsilon}^{-\frac{1}{3}} l_n^{\frac{2}{3}} \left(\frac{l_n}{l_0}\right)^{\frac{1}{3}(3-D)},$$

$$E_n \sim \bar{\epsilon}^{\frac{2}{3}} l_n^{\frac{2}{3}} \left(\frac{l_n}{l_0}\right)^{\frac{1}{3}(3-D)},$$

$$E(k) \sim \bar{\epsilon}^{\frac{2}{3}} k^{-\frac{5}{3}} \left(k l_0\right)^{-\frac{1}{3}(3-D)}.$$

Now we can see the intermittency correction have been expressed in terms of a fractal dimension D. The spectra are slightly steepened. By fitting the spectral data, we are able to deduce the effective/fractal dimension, which is $D = 2.7 - 2.8$.

Remark: The mean dissipation rate is still classical Kolmogorov result

$$\bar{\epsilon} \sim \frac{v_0^3}{l_0},$$

which is not affected by the intermittency parameter D. In contrast, the dissipation scale l_d is affected. By equating the eddy turnover time t_n to viscous diffusion time ${l_n^2}/{\nu}$, we obtain the dissipation scale

$$l_d \sim l_0 R_e^{-\frac{3}{1+D}},$$

where $R_e = {l_0 v_0}/{\nu} \sim \bar{\epsilon}^{\frac{1}{3}} l_0^{\frac{4}{3}} \nu^{-1}$ is the Reynolds number. If $D = 3$, then $l_d \sim {\nu^{3/4}}/{\epsilon^{1/4}}$, which recovers the K41 results.

IV. HIGHER-ORDER STATISTICS

We now turn to higher-order structure functions. The non-dimensionalized moments of higher orders are defined as

$$S_p = \frac{\langle (\delta v)^p \rangle}{\left\langle (\delta v)^2 \right\rangle^{p/2}}.$$

In K41 theory $\delta v(l) \sim \epsilon^{1/3} l^{1/3}$, then $\langle (\delta v)^p \rangle \sim \epsilon^{p/3} l^{p/3}$, so the normalized higher order moments $S_p \sim 1$. The K41 energy spectral scalings are supported by the experimental results, although small corrections can be added the exponent, larger departures due to intermittency can be seen in higher-order statistics. Hence the higher-order moments are more severe probes of small scale structures of turbulence than energy is.

Two important higher order moments are skewness S $(p = 3)$ and kurtosis K $(p = 4)$. Skewness is a measure of asymmetry of the probability distribution function (PDF), and is relevant to the turbulent energy transfer, since

$$\partial_t \left\langle v^2 \right\rangle \sim \left\langle v^3 \right\rangle \sim S.$$

$S\left(\nabla v\right)$ is also a measure of vortex stretching:

$$\left\langle (\nabla v)^3 \right\rangle \sim \int k^2 T(k) dk$$

$$\left\langle (\nabla v)^2 \right\rangle \sim \int k^2 E(k) dk$$

$$S \sim \frac{\int k^2 T(k) dk}{\left(\int k^2 E(k) dk\right)^{3/2}},$$

where the $\int k^2 T(k) dk$ is the time-rate-change of the enstrophy due to nonlinear interactions. Kurtosis represents the flatness of the distribution, a PDF with heavier tails will have larger kurtosis. A Gaussian distribution have $K = 3$. Deviations from $K = 3$ typically indicate long-range correlations and non-Gaussian behavior.

In β-model, eddies fill only a fraction β_n of the space, we have

$$\begin{aligned}\left\langle \left(\delta v_n\right)^p \right\rangle &\sim \beta_n \left\langle \left(\delta v_n\right)^p \right\rangle_{cond} \\ &\sim \beta_n v_n^p \\ &\sim \bar{\epsilon}^{\frac{p}{3}} l_n^{\frac{p}{3}} \left(\frac{l_n}{l_0}\right)^{\zeta_p},\end{aligned}$$

where $\zeta_p = \frac{1}{3}(3 - D)(3 - p)$. The dimensionless structure functions are given by

$$S_p\left(l_n\right) \sim \left(\frac{l_n}{l_0}\right)^{\xi_p},$$

where

$$\xi_p = \zeta_p - \frac{1}{2} p \zeta_2 = \frac{1}{2}(3 - D)(2 - p).$$

We can see that $\xi_p < 0$, when $p > 2$. This means the structures grow stronger on small scales. In particular, we obtain the the skewness

$$S \sim R_e^{3(3-D)/2(1+D)},$$

and kurtosis

$$K \sim S^2.$$

Note:

- the departures from K41 are strongest at smallest scales.
- β-model shows stages in cascade have "memory" of initial scale l_0, since it appears explicitly in the formulas. Hence the β-model is sensitive to initial conditions, which conforms to experimental observations. This property is lacking in K41 model.
- the effective dimension is $D \approx 2.8$ from the data fitting. This means the dissipative structures are homogeneous fractals, i.e. highly convoluted sheets, distributed in 3-dimension space, which do not uniformly fill the embedding space.
- but ζ_p departs β-model at high p, as shown in figure 5. Multi-fractal model are then introduced to resolve this [3, 4]. The idea of multi-fractal model is that the velocity field has no single exponent h (single dissipative structure) such that $\delta v_l \sim l^h$, but a continuous spectrum of exponents (multiple dissipative structures). In other words, in the inertial range one has

$$\delta v_l(\mathbf{x}) \sim l^h,$$

  where $\mathbf{x} \in S_h$, and S_h is a fractal set with dimension $D(h)$ and $h_{min} < h < h_{max}$. This is known as the *local scale-invariance*, in contrast to the global scale-invariance in K41 and β-model.

The fluctuating "dissipation" at point $\mathbf{r}$ is defined as

$$\bar{\epsilon}(\mathbf{r}) \sim \nu (\nabla v)^2 .$$

Clearly $\epsilon(\mathrm{r})$ depends on the dissipation range, in that sense its primary contribution comes from the smallest scale $\sim l_d$. Then in β-model we have

$$\begin{aligned}
\left\langle \tilde{\epsilon}^2 \right\rangle &\sim \nu^2 \left\langle (\nabla v)^2 (\nabla v)^2 \right\rangle \\
&\sim \nu^2 \beta\left(l_d\right) \frac{\tilde{v}^4\left(l_d\right)}{l_d^4}, \\
\frac{\left\langle \tilde{\epsilon}^2 \right\rangle}{\left\langle \bar{\epsilon}^2 \right\rangle} &\sim R_e^{3\frac{3-D}{1+D}} \\
&\sim K.
\end{aligned}$$

The dissipation correlation function is defined as

$$\langle \epsilon(\mathbf{r}) \epsilon(\mathbf{r}+\mathbf{l}) \rangle \sim \left\langle \epsilon^2 \right\rangle \text{Prob}\left\{ \mathbf{r} \text{ and } \mathbf{r}+\mathbf{l} \text{ belong to same eddy} \right\} .$$

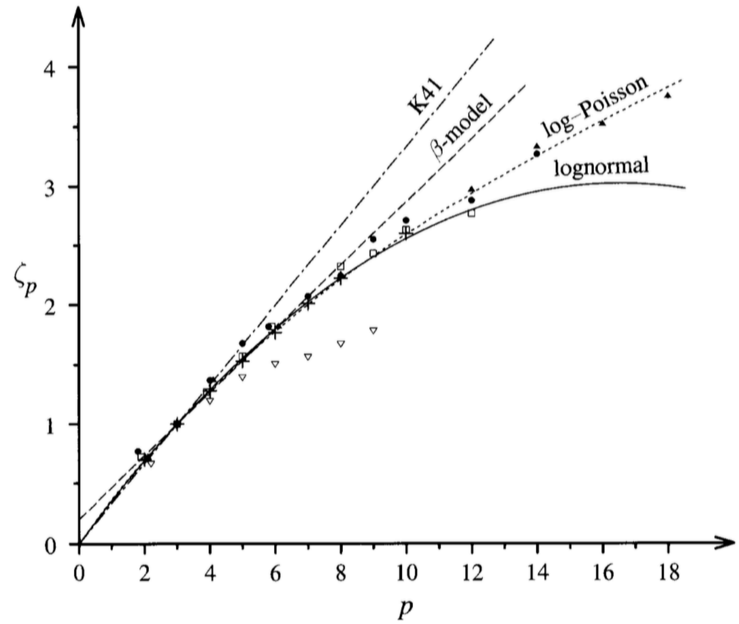

Figure 5. Structure function scaling exponents, ζ_p, vs p. Markers are experimental data and lines are model fittings.

If two points are correlated by an eddy l_n, then they are correlated by all larger eddies in same region. The energy transfer per unit mass from m-eddies to $(m+1)$-eddies is

$$\left\langle \epsilon \left(l_m \right) \right\rangle \sim \frac{v^3 \left(l_m \right)}{l_m}.$$

We thus obtain the correlation function by sum up all the scales, and the largest value comes from $m = n$, so that the final result is

$$\begin{aligned} \langle \epsilon (\mathbf{r}) \, \epsilon (\mathbf{r} + \mathbf{l}) \rangle &\sim \sum_{m=0}^{n} \beta_m \left(\frac{v_m^3}{l_m} \right)^2 \\ &\sim \bar{\epsilon}^2 \left(\frac{l}{l_0} \right)^{-(3-D)} . \end{aligned}$$

This expression, which is singular at $l \to 0$, indicates a very strong correlation in dissipation at small scales.

V. TIME SERIES AND HURST EXPONENT

So far we have introduced the fractals and the β-model, which are mostly focused on intermittency in space. However, the time series also exhibits the fractal nature, technically, self-affine property. For example, given a Brownian time series $B(t)$, then the expectation of first and second order are:

$$\mathrm{E}\left\{B(t+T)-B(t)\right\}=0,$$
$$\mathrm{E}\left\{(B(t+T)-B(t))^2\right\}=T,$$

where the increment $T > \tau_{ac}$, since the events in this random motion are uncorrelated. Then we can generalize it by introduce an exponent without the restriction to Brownian motions:

$$\mathrm{E}\left\{B_H(t+T)-B_H(t)\right\}=0,$$
$$\mathrm{E}\left\{\left(B_H(t+T)-B_H(t)\right)^2\right\}=T^{2H},$$

where H is named the Hurst exponent which is a measure of the roughness of a time series. Clearly, when $H \neq 0.5$, $B(t)$ will deviate from the Gaussian behavior, which means the correlation between events becomes non-negligible. Actually, the Hurst exponent is the counterpart of fractal dimension for intermittency in time series.

Studies involving the Hurst exponent were originally developed by British hydrologist H. E. Hurst for the practical matter of determining optimum dam sizing for the Nile river. He discovered that the rescaled range $^R/_S$, which represents rescaled ideal capacity of a reservoir, is proportional to k^H for large k, where k is the time span and $H > 0.5$. This empirical finding was in contradiction to results for the Brownian motion and random walk process. For any stationary process with short-range dependence, $^R/_S$ should behave asymptotically $\sim k^{0.5}$.

Later, Mandelbrot related the Hurst exponent to fractal geometry and used it as a measure of the long-term memory of a time series. In addition, Mandelbrot also brought in two terms called "Joseph" effect and "Noah" effect:

- The "Joseph" effect says a time series has long periods with high levels, and then followed by long periods with low levels. The series looks stationary overall. At short time periods, it seem to have cycles and local trends. But at the whole scale, there is no apparent persisting cycle or global trend. This effect infers $H > 0.5$. The name refers to the Old Testament story that Egypt would experience seven years of rich harvest followed by seven years of famine.

- The "Noah" effect says a time series tends to have large and concentrated events, indicating an infinite variance. The name is derived from the biblical story of the Great Flood.

Technically, the Hurst exponent, H, is defined in terms of the asymptotic behavior of the rescaled range as a function of the time span of a time series as follows:

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{R(n)}{S(n)}\right] = Cn^H,$$

where $R(n)$ is the range of the first n values, $S(n)$ is their standard deviation. As mentioned above, the H exponent encodes fractal character of dynamics, i.e.

$$D_0 \sim \frac{\ln N}{\ln\left({}^1\!/\!_\epsilon\right)} \leftrightarrow H \sim \frac{\ln \Delta B}{\ln \Delta t},$$

suggesting a direct relation to the fractal dimension. Using the Hurst exponent we can classify time series into types and gain some insight into their dynamics. Here are some types of time series and the Hurst exponents associated with each of them.

A. Brownian Time Series ($H = 0.5$)

In a Brownian time series (also known as random walk), as shown in figure 6, observations are uncorrelated in time – being higher or lower than the current observation are equally possible for a later observation. A Hurst exponent close to 0.5 is indicative of a Brownian time series.

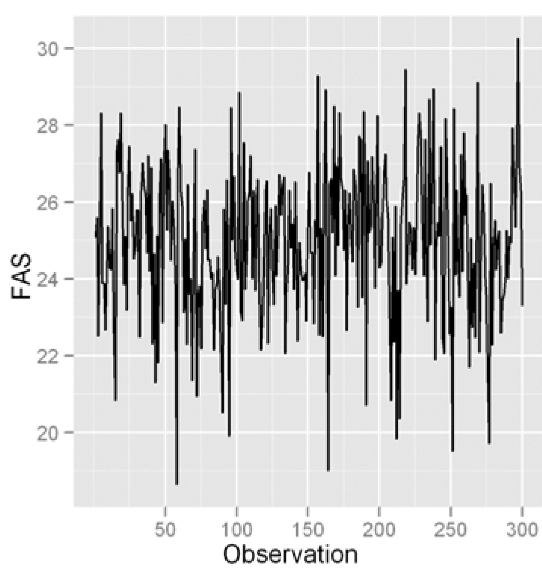

Figure 6. A Brownian time series ($H = 0.53$).

B. Anti-Persistent Time Series ($0 < H < 0.5$)

In an anti-persistent time series (also known as a mean-reverting series) an increase will most likely be followed by a decrease or vice-versa (i.e., values will tend to revert to a mean), as shown in figure 7. This means that movements have a tendency to return to a long-term mean. A Hurst exponent value between 0 and 0.5 is indicative of anti-persistent behavior and the closer the value is to 0, the stronger is the tendency for the time series to revert to its long-term mean value.

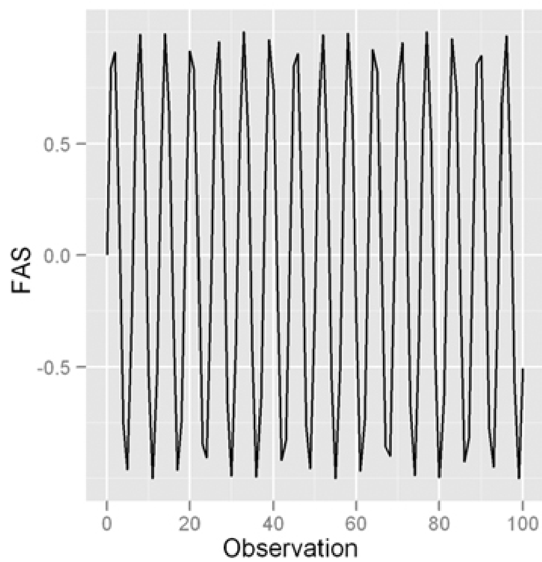

Figure 7. An anti-persistent time series ($H = 0.043$).

C. Persistent Time Series ($0.5 < H < 1$)

In a persistent time series an increase in values will most likely be followed by an increase in the short term, and a decrease in values will most likely be followed by another decrease in the short term. The figure 8 provides an example of a persistent time series. A Hurst exponent value between 0.5 and 1.0 indicates persistent behavior; the larger the H value, the stronger the trend.

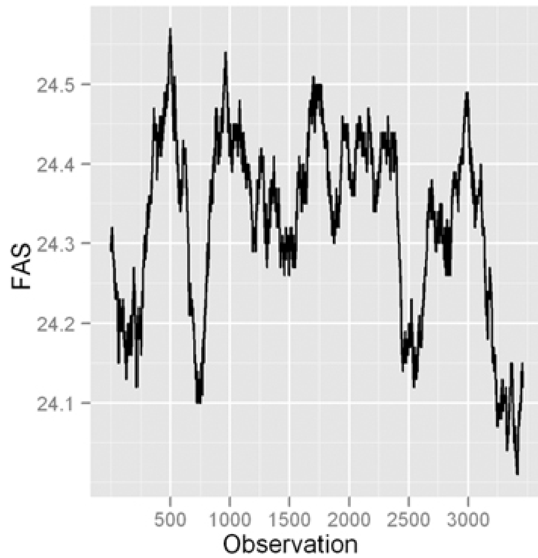

Figure 8. A persistent time series ($H = 0.95$).

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